Max-Flow Min-Cut Theorem and Faster Algorithms in a Circular Disk Failure Model

Yusuke Kobayashi  
University of Tokyo  
Tokyo, 113-8656, Japan  
kobayashi@mist.i.u-tokyo.ac.jp

Kensuke Otsuki  
University of Tokyo  
Tokyo, 113-8656, Japan  
kensuke_otsuki@mist.i-u-tokyo.ac.jp

Abstract—Fault-tolerance is one of the most important factors in designing networks. Failures in networks are sometimes caused by an event occurring in specific geographical regions such as hurricanes, earthquakes, bomb attacks, and Electromagnetic Pulse (EMP) attacks. In INFOCOM 2012, Neumayer et al. introduced geographical variants of max-flow min-cut problems in a circular disk failure model, in which each failure is represented by a disk with a predetermined size. In this paper, we solve two open problems in this model: we give a first polynomial-time algorithm for the geographic max-flow problem, and prove a conjecture of Neumayer et al. on a relationship between the geographic max-flow and the geographic min-cut.

I. INTRODUCTION

Fault-tolerance is one of the most important factors in designing networks. In most studies on fault-tolerance in networks, “connectivity” of the network is regarded as the measure of robustness (e.g. [1]–[3]). However, failures in networks are sometimes caused by an event occurring in specific geographical regions such as hurricanes, earthquakes, bomb attacks, and Electromagnetic Pulse (EMP) attacks. Recently, some models in which such localized failures are taken into consideration are proposed in [4]–[10].

In particular, Neumayer et al. [5] considered the model in which each failure is represented by a hole (disk) with a predetermined size (see Section II for details), and they gave a polynomial-time algorithm for computing the minimum number of failures that disconnect two specified nodes \(s\) and \(t\), which they call the “geographic min-cut”. They also formulated the problem of finding the maximum number of \(s\)-\(t\) paths such that no two paths can be disconnected by the same hole, which they call the “geographic max-flow”. Similar problems are considered in [11], [12].

In this paper, we further develop theory and algorithms for geographic min-cut and geographic max-flow in this model. Our contributions are described as follows.

- We give a min-max theorem that characterizes a geographic max-flow (Theorem 1).
- We show that the geographic min-cut is at most the geographic max-flow plus one (Theorem 3), which was conjectured by Neumayer et al.
- We give a first polynomial-time algorithm for the geographic max-flow (Theorem 4).

We also note that we can extend our results to the case when each hole is of different shapes.

We emphasize here that the second and the third results solve two open problems raised by Neumayer et al. [5] in INFOCOM 2012.

II. PROBLEM SETTINGS

Let \(G = (V, E)\) be a graph drawn in the plane with a node set \(V\), a link set \(E\), and two distinct nodes \(s, t \in V\), where each link is drawn as a line segment. In what follows, a link is sometimes called an edge. Let \(r_i\) be a hole radius and \(r_p(> r_b)\) be a protection radius. A disk of radius \(r_p\) whose center is \(s\) or \(t\) is called a protective disk. Define \(H(r_b, r_p)\) as the set of all disks of radius \(r_b\) whose centers are not contained in protective disks of radius \(r_p\). Each element of \(H(r_b, r_p)\) is called a hole in this paper.

We consider a geographic variant of the min-cut problem, which is defined as follows.

Geographical Min-Cut by Circular Disasters (GMCCD)

**Input:** a graph \(G = (V, E)\) drawn in the plane, two distinct nodes \(s\) and \(t\), a hole radius \(r_b\), and a protection radius \(r_p(> r_b)\).

**Find:** a minimum cardinality set of holes in \(H(r_b, r_p)\) that disconnect \(s\) from \(t\).

Let \(\text{MIN-CUT}\) denote the optimal value of this problem. A set of holes in \(H(r_b, r_p)\) that disconnect \(s\) from \(t\) is called a hole cut in this paper. We can also consider a geographic variant of the max-flow problem.
Geographical Max-Flow by Circular Disasters (GMFCD)

Input: a graph $G = (V, E)$ drawn in the plane, two distinct nodes $s$ and $t$, a hole radius $r_h$, and a protection radius $r_p(> r_h)$.

Find: a maximum cardinality set of $s$-$t$ paths such that no hole in $\mathcal{H}(r_h, r_p)$ intersects a pair of these paths.

Let MAX-FLOW denote the optimal value of this problem. So far, no polynomial-time algorithm for GMFCD was known, whereas a polynomial-time algorithm for GMCCD was given in [5].

Example 1. Consider the graph as in Fig. 1 (which is an example given in [5]). Small circles represent holes of radius $r_h$ and two large shaded (or yellow in the color version) circles are protective disks. In the graph, we can easily see that MAX-FLOW = 1 and MIN-CUT = 2.

III.地理Max-Flow MIN-CUT定理


A. Statement of the theorem

Let $C$ be a closed curve in the plane that does not go through $s$ or $t$. We define the winding number $w(C)$ of $C$ as the number of times that $C$ separates $s$ and $t$. More precisely, let $L$ be the line segment connecting $s$ and $t$, and fix orientations of $L$ and $C$. Let $w_1(C)$ be the number of times that $C$ crosses $L$ from left to right, and $w_2(C)$ be the number of times that $C$ crosses $L$ from right to left. Then, define $w(C) := |w_1(C) − w_2(C)|$.

We say that a closed curve $C$ can be represented as an alternating curve of length $l$ if it is a concatenation of $J_1, L_1, J_2, L_2, \ldots, J_l, L_l$ in this order such that

- $J_1, J_2, \ldots, J_l$ are curves each contained in a face of $G$, and
- for each $i = 1, 2, \ldots, l$, $L_i$ is a line segment that can be covered by a hole in $\mathcal{H}(r_h, r_p)$.

Note that $J_i$ and $L_i$ might be a single point. For a closed curve $C$, let $l(C)$ be the minimum number $l$ such that $C$ can be represented as an alternating curve of length $l$.

By using these notations, we can give an exact min-max theorem for MAX-FLOW, whose proof is given in the next subsection.

Theorem 1. Suppose we are given a graph $G = (V, E)$ drawn in the plane, two distinct nodes $s$ and $t$, a hole radius $r_h$, and a protection radius $r_p(> r_h)$. Then,

\[
\text{MAX-FLOW} = \min \left\{ \frac{l(C)}{w(C)} \mid C \text{ is a closed curve} \right\}.
\]

Example 2. Consider again the graph as in Fig. 1. As shown in Fig. 2, there exists a closed curve $C$ such that $l(C) = 3$, $w(C) = 2$, and hence $\frac{l(C)}{w(C)} = 1$. This value is equal to MAX-FLOW.

B. Proof of Theorem 1

In this subsection, we give a proof of Theorem 1. Our proof is based on ideas in [13] (see also [14], [15]), which shows a min-max theorem for maximum induced disjoint $s$-$t$ paths in plane graphs.

We say that two $s$-$t$ paths are separated, if no hole in $\mathcal{H}(r_h, r_p)$ intersects both of these paths. For a pair of edges $e, e' \in E$, if there exists a hole in $\mathcal{H}(r_h, r_p)$ that intersects both edges, then we take two points $w_{e, e'}$ on $e$ and $w_{e', e}$ on $e'$ that are contained in the common hole arbitrarily. Define

\[ W := \{w_{e, e'} \mid e, e' \in E \text{ contained in a common hole}\}. \]

Let $L$ be the set of all line segments with both endpoints in $W$ such that each line segment is contained in a hole in $\mathcal{H}(r_h, r_p)$. Note that a line segment might be a single point, that is, $(w, w') \in L$ for $w \in W$. Then, two $s$-$t$ paths $P$ and $P'$ are not separated if and only if there exists a line segment $(w, w') \in L$ such that $w$ is on $P$ and $w'$ is on $P'$. Theoretically, $|W|$ is bounded by $|E|^2$ which is a polynomial size. Practically, we can obtain $W$ by adding a small number of points to $V$, because we have to add points to $V$ only in some exceptional cases (see e.g. Fig. 3).

First, we show

\[ \text{MAX-FLOW} \leq \min \left\{ \frac{l(C)}{w(C)} \mid C \text{ is a closed curve} \right\}. \]

Suppose we have $s$-$t$ paths $P_1, \ldots, P_k$ that are pairwise separated and let $\tilde{C}$ be a closed curve that is a concatenation of $J_1, L_1, J_2, L_2, \ldots, J_l(C), L_l(C)$ in this order. Since $C$ intersects
each $P_i$ at least $w(C)$ times, each $P_i$ intersects at least $w(C)$ line segments of $L_1, L_2, \ldots, L_{l(C)}$. This means that $l(C) \geq k \cdot w(C)$. By the integrality of $k$, we have $k \leq \left\lfloor \frac{l(C)}{w(C)} \right\rfloor$.

Next we show

\[
\text{MAX-FLOW} \geq \min \left\{ \frac{l(C)}{w(C)} \mid C \text{ is a closed curve} \right\}
\]

by giving an algorithm for finding either $k$ pairwise separated $s$-$t$ paths or a closed curve $C$ with $\frac{l(C)}{w(C)} < k$ for any $k$.

For any $s$-$t$ path $P$, we suppose that it is oriented from $s$ to $t$ and $P^{-1}$ is oriented from $t$ to $s$. For two $s$-$t$ paths $P'$ and $P''$ without crossings, let $R(P', P'')$ denote the closed region encircled by $P', (P'')^{-1}$ in clockwise orientation, where $P'$, $(P'')^{-1}$ is the closed curve obtained by concatenating $P'$ and $(P'')^{-1}$. For two $s$-$t$ paths $P'$ and $P''$ without crossings, a pair $(P', P'')$ is \textit{clockwise separated} if for any hole $H$ in $\mathcal{H}(r_b, r_p)$, $R(P', P'') - H$ is connected. Obviously, $P'$ and $P''$ are separated if and only if a pair $(P', P'')$ and a pair $(P'', P')$ are both clockwise separated (see Fig. 4 for an example). In what follows, we show the inequality by the induction on $k$.

1) Induction step: First, we consider the case $k \geq 3$ under the assumption that we have $k-1$ pairwise separated $s$-$t$ paths $P_1, \ldots, P_{k-1}$. We may assume that these paths do not cross each other, and the first edges of $P_1, \ldots, P_{k-1}$ occur in this order clockwise at $s$. Let $P_k$ be an $s$-$t$ path in $R(P_{k-1}, P_1)$ such that $(P_{k-1}, P_k)$ is clockwise separated. In our algorithm, we start with the $k$ paths $P_1, \ldots, P_{k-1}, P_k$, and we replace one path with a new path, repeatedly. Our algorithm is described in Algorithm 1 (see Fig. 5 for an example).

If $(P_1, P_{l-k+1})$ is clockwise separated in line 2 of Algorithm 1, then $P_{l-k+1}, \ldots, P_{l-1}, P_l$ are pairwise separated.

\footnote{This condition is equivalent to “separated” when $k \geq 3$. Since we will use the same argument for the case of $k = 2$ later, we use the condition “clockwise separated” here.}
Algorithm 1 Induction step

Input: pairwise separated $k - 1$ $s$-$t$ paths $P_1, \ldots, P_{k-1}$ and an $s$-$t$ path $P_k$ in $R(P_{k-1}, P_1)$ such that $(P_{k-1}, P_k)$ is clockwise separated

Output: pairwise separated $k$ $s$-$t$ paths or a closed curve $C$ with $\frac{l(C)}{w(C)} < k$

1: for $l = k, k + 1, \ldots, k + |W| + 1$ do
2: \hspace{1em} if $(P_1, P_{l-1})$ is clockwise separated\footnote{If $(P_1, P_{l-1})$ is not clockwise separated, we can reroute $P_1$ along the border of a face containing $v_N$, then the obtained path contains a node $v_{N-l}$ which is close to a node $v_{N-l-1}$ in $P_{N-l-1}$. Therefore, $(w_{N-l-1}, v_{N-l-1}) \in \mathcal{L}$, and there exists a curve from $v_{N-l-1}$ to $v_{N-l}$ contained in a face of $G$.}
3: \hspace{2em} return $P_{l-1}, P_{l-2}, \ldots, P_1$ that are separated paths
4: \hspace{2em} end
5: \hspace{1em} let $P_{l+1}$ be the $s$-$t$ path in $R(P_{l-1} + 1, P_{l-1} + 2)$ such that $(P_{l-1}, P_{l+1})$ is clockwise separated and $R(P_{l+1} + 1, P_{l+1} + 2)$ is maximized under this condition
6: \hspace{2em} end
7: return a closed curve $C$ with $\frac{l(C)}{w(C)} < k$

Fig. 6. Definition of $w_{N-l-1}$ and $v_{N-l-1}$. If we reroute $P_N$ along the border of a face containing $v_N$, then the obtained path contains a node $v_{N-l}$ which is close to a node $v_{N-l-1}$ on $P_{N-l-1}$. Therefore, $(w_{N-l-1}, v_{N-l-1}) \in \mathcal{L}$, and there exists a curve from $v_{N-l-1}$ to $v_{N-l}$ contained in a face of $G$.

Let $N := k + |W| + 1$. By the assumption, $(P_1, P_{l+1})$ is clockwise separated for $i = l - k + 1, \ldots, l - 1$. Therefore, it suffices to return $P_{l+1}, P_{l+2}, \ldots, P_1$. In what follows, we give a procedure for finding a closed curve $C$ with $\frac{l(C)}{w(C)} < k$ (line 8) when such paths do not appear while $l = k, k + 1, \ldots, k + |W| + 1$.

If we have a hole in $\mathcal{H}(r_b, r_p)$ that disconnects $s$ from $t$, then we have a closed curve $C$ with $l(C) = w(C) = 1$. Otherwise, we have an $s$-$t$ path $P$ in $G$ as in the second case of Lemma 2. In this case, define $P_1 = P_2 = P$ and assume that $P_2$ is to the left of $P_1$. Then, since $(P_1, P_2)$ is clockwise-separated, we can apply Algorithm 1 to obtain pairwise separated two $s$-$t$ paths or a closed curve $C$ with $\frac{l(C)}{w(C)} < 2$, which completes the proof for the case of $k = 2$.

C. Proof of the conjecture

By using Theorem 1, in this subsection we give a proof of the conjecture of Neumayer et al.

Theorem 3. Suppose we are given a graph $G = (V, E)$ drawn in the plane, two distinct nodes $s$ and $t$, a hole radius $r_b$, and a protection radius $r_p (> r_b)$. Then,

$$\text{MAX-FLOW} \leq \text{MIN-CUT} \leq \text{MAX-FLOW} + 1.$$
of Theorem 1 in Section III-B, we obtain a polynomial-time algorithm for computing MAX-FLOW.

Theorem 4. An optimal solution of the GMFCD and a closed curve $C$ minimizing $\frac{\ell(C)}{w(C)}$ can be computed in polynomial time.

Note that this is the first polynomial-time algorithm for the GMFCD. The most time consuming part is Algorithm 1 that runs in $O(|W|^2)$ time. Since we execute Algorithm 1 at most $k := \text{MAX-FLOW}$ times, the total running time is $O(k|W|^2)$. In most practical cases, since $k$ is small and $|W| = O(|V|)$, the running time is $O(|V|^2)$.

Next, we propose a new algorithm for the GMCCD, which is simpler (and probably faster) than known algorithms.

Theorem 5. An optimal solution of the GMCCD can be computed in polynomial time.

Proof: By Theorem 4, we can compute $s$-$t$ paths $P_1, \ldots, P_k$ that are pairwise separated, where $k := \text{MAX-FLOW}$. Furthermore, by Theorem 3, we can also obtain a hole cut of size $k + 1$ (i.e., a set of $k + 1$ holes in $\mathcal{H}(r_b, r_p)$ that disconnect $s$ from $t$). Since MAX-FLOW $\leq$ MIN-CUT, our remaining task is to find a hole cut of size $k$ if one exists.

Since the case of $k = 1$ is easy, in what follows we suppose $k \geq 2$. We may assume that $P_1, \ldots, P_k$ do not cross each other, and the first edges of $P_1, \ldots, P_k$ occur in this order clockwise at $s$. Recall that $R(P_{i-1}, P_i)$ is the closed region encircled by the closed curve $P_{i-1} \cup P_i^{-1}$ in clockwise orientation. For $i = 1, \ldots, k$, let $\mathcal{F}_i$ be the set of all faces of $G$ contained in $R(P_{i-1}, P_i)$, where $P_0 := P_k$. We observe that a hole cut of size $k$ exists if and only if there exists a closed curve $C$ with $w(C) = 1$ and $l(C) = k$ that is represented as a concatenation of $J_1, L_1, J_2, L_2, \ldots, J_k, L_k$ in this order, where $J_i$ is a curve contained in a face of $\mathcal{F}_i$, and $L_i \in \mathcal{L}$ is a line segment connecting $R(P_{i-1}, P_i)$ and $R(P_i, P_{i+1})$. Note that $P_{k+1} := P_1$ and $\mathcal{F}_{k+1} := \mathcal{F}_1$. To check the existence of such a curve, we construct a directed graph $D = (\mathcal{F}, A)$, where

$$\mathcal{F} := \bigcup_i \mathcal{F}_i$$

$$A := \{(\mathcal{F}_i, \mathcal{F}_{i+1}) \mid i \in \{1, \ldots, k\}, \mathcal{F}_i \in \mathcal{F}_i, \mathcal{F}_{i+1} \in \mathcal{F}_{i+1}, \exists H \in \mathcal{H}(r_b, r_p) \text{ intersecting } \mathcal{F}_i, P_i, \text{ and } F_{i+1}\}$$

Then, finding a hole cut of size $k$ is equivalent to finding a directed cycle of length $k$ in $D$.

With this observation, we can compute MIN-CUT by Algorithm 2, and it is obvious that it runs in polynomial time.

V. EXPERIMENTAL RESULTS

In this section, we describe experimental results. We implemented our algorithms for the GMFCD and the GMCCD, and evaluated their performance by computational experiments. Our experiments were conducted on a computer with Intel Core i7, 2.8 GHz and 8 GB of memory. All programs are written in Java.
Algorithm 2 Find-MIN-CUT

Input: pairwise separated k s-t paths $P_1, \ldots, P_k$ and a hole cut of size $k + 1$

Output: a hole cut of minimum size

1: construct the directed graph $D = (F, A)$ defined as (*)
2: for all $F \in F_1$ do
3: by a breadth-first-search from $F$, try to find a directed cycle of length $k$ in $D$ containing $F$
4: if such a directed cycle exists then
5: return a hole cut of size $k$ corresponding to the directed cycle
6: end if
7: end for
8: return a hole cut of size $k + 1$

Fig. 10. Experimental result (MAX-FLOW)

As we have seen before, our algorithm for computing MAX-FLOW consists of the induction step (Algorithm 1) and the base cases (Lemma 2). Practically, since most short s-t paths satisfy the second condition of Lemma 2, we do not need an implementation of the algorithm in Lemma 2. Therefore, we can compute MAX-FLOW by just applying Algorithm 1, repeatedly. We generated input plane graphs with 1000 nodes randomly in a $300 \times 400$ rectangular and applied our algorithm to them. Then we can solve the GMFCD in a few seconds. As an example, a computational result with 1000 nodes and 7 paths is shown in Fig. 10, where we set $r_b = 10$ and $r_p = 30$. Note that we regarded the nodes on the boundary of the protective disks as the terminals to make the figure easier to see.

We also implemented Algorithm 2, and applied it to randomly generated graphs. For graphs with 1000 nodes, Algorithm 2 computes MIN-CUT in a few seconds. Fig. 11 is a computational result with 1000 nodes, where we set $r_b = 7$ and $r_p = 30$. In this case, we can see MIN-CUT = MAX-FLOW = 8.

Fig. 11. Experimental result (MIN-CUT)

VI. CONCLUDING REMARKS

In this paper, we discussed the geographical min-cut and the geographical max-flow in the model, in which every hole is a disk of the same radius $r_b$. We proved a min-max theorem and gave polynomial-time algorithms for the GMFCD and the GMCCD that can be applied to large graphs.

Our results can be extended to the case with holes of different shapes. Suppose that $\mathcal{H}$ is a set of convex shapes (holes) satisfying the following property.

Property: Suppose that two line segments $L_1$ in $H_1 \in \mathcal{H}$ and $L_2$ in $H_2 \in \mathcal{H}$ are crossing. Then, $H_1$ also contains an endpoint of $L_2$ or $H_2$ also contains an endpoint of $L_1$. (See the case analysis of the proof of Theorem 3.)

In Theorems 1 and 3, we can replace $\mathcal{H}(r_b, r_p)$ with any set $\mathcal{H}$ satisfying the above property. For example, $\mathcal{H}$ can be a set of disks of different sizes or a set of axis parallel squares. In particular, by setting $\mathcal{H}$ as the set of all edges (not incident to $s$ and $t$), we obtain a min-max theorem for the maximum induced disjoint s-t paths [13] as a special case of Theorem 1.

Note that we cannot extend our proofs to the case when $\mathcal{H}$ is a set of general connected shapes violating the property above. Actually, computing MAX-FLOW becomes NP-hard when $\mathcal{H}$ does not necessarily satisfy the property [11].

ACKNOWLEDGEMENTS

The authors thank anonymous reviewers for helpful comments. This work is supported by JST, ERATO, Kawarabayashi Large Graph Project, Japan. The first author is supported by Grant-in-Aid for Scientific Research, MEXT, Japan.

REFERENCES

APPENDIX

In the appendix, we give a proof of Lemma 2, which we restate here.

Lemma. Suppose that $s$ and $t$ are connected in $G$. We can find in polynomial time either

- a hole in $\mathcal{H}(r_b, r_p)$ that disconnects $s$ from $t$, or
- an $s$-$t$ path $P$ in $G$ such that for any line segment $L \in \mathcal{L}$, the union of $P$ and $L$ (which is regarded as a subset of the plane) contains no closed curve separating $s$ and $t$.

Proof: To simplify the argument, we discuss the case when $\mathcal{L}$ is replaced with $\mathcal{L}'$, where $\mathcal{L}'$ is the set of all line segments contained in holes in $\mathcal{H}(r_b, r_p)$. Note that there is no essential difference between $\mathcal{L}$ and $\mathcal{L}'$.

Assume that $G$ contains no $s$-$t$ path satisfying the condition in the second case. That is, for any $s$-$t$ path $P$, there exists a line segment $L \in \mathcal{L}'$ such that $P \cup L$ contains a closed curve separating $s$ and $t$.

For a path $P$ and for two points $x, y \in P$, the part of $P$ between $x$ and $y$ is denoted by $P[x, y]$. For an $s$-$t$ path $P$ and for two points $x, y \in P$, we denote $x \prec_P y$ if $x$ appears earlier than $y$ when we move from $s$ to $t$ along $P$.

For each $s$-$t$ path $P$, we take a line segment $L_P \in \mathcal{L}'$ connecting $p, q \in P$ with $p \prec_P q$ satisfying the following conditions (see Fig. 12).

1. $P[p, q] \cup L_P$ forms a closed curve separating $s$ and $t$.
2. Subject to (1), $q$ is maximal with respect to $\prec_P$.
3. Subject to (1) and (2), $p$ is minimum with respect to $\prec_P$.

Among all $s$-$t$ paths, we choose an $s$-$t$ path $P$ such that $P[q, t]$ is maximal (i.e. it is as long as possible).

We show the following claim, which will be used to reroute the path $P$.

Claim 6. Let $P$ be a path chosen as above. If there is no hole in $\mathcal{H}(r_b, r_p)$ that disconnects $s$ from $t$, then there exists a path $Q$ from $s$ to a node $t'$ on $P[p, q] \setminus \{p, q\}$ with the following properties:

- $Q$ does not intersect with $P[p, q] \cup L_P$ except its endnodes (i.e., $Q$ is contained in the inside of cycle $P[p, q] \cup L_P$), and
- there exists no line segment $L \in \mathcal{L}'$ connecting $x \in P[p, q]$ and $y \in Q$ such that $x \prec_P t'$ and $P[x, t'] \cup Q[y, t'] \cup L$ forms a closed curve separating $s$ and $t$ (see Fig. 13).

Proof: Suppose that there is no hole in $\mathcal{H}(r_b, r_p)$ that disconnects $s$ from $t$. This implies that there exists a path $Q$ from $s$ to $P[p, q] \setminus \{p, q\}$ such that $Q$ does not intersect with $P[p, q] \cup L_P$ except its endnodes.
In order to derive a contradiction, we assume that every path \( Q \) from \( s \) to \( P[p, q] \setminus \{p, q\} \) violates the second condition, that is, for each \( Q \), there exists a line segment \( L_Q \in \mathcal{L} \) connecting \( x \in P[p, q] \) and \( y \in Q \) such that \( x \prec_p t' \) and \( P[x, t'] \cup Q[y, t'] \cup L_Q \) forms a closed curve separating \( s \) and \( t \), where \( t' \) is the endnode of \( Q \) on \( P[p, q] \setminus \{p, q\} \). Among such line segments, we choose \( L_Q \) as the shortest one. (If there exist more than one shortest line segments, we choose \( L_Q \) so that \( y \) is as small as possible with respect to \( \prec_p \)). Among all paths from \( s \) to \( P[p, q] \setminus \{p, q\} \), we choose a path \( Q \) such that \( L_Q \) is as long as possible.

Since we assume that there is no hole separating \( s \) and \( t \), there exists a path \( R \) from \( s \) to \( (P[x, t'] \cup Q[y, t']) \setminus \{x, y\} \) that does not intersect with \( L_Q \). Let \( r_2 \) be the endpoint of \( R \) that is different from \( s \), and let \( r_1 \) be the point in \( Q \cap R \) such that no inner nodes of \( R[r_1, r_2] \) are contained in \( Q \). Define the path \( Q' \) as \( Q' := Q[s, r_1] \cup R[r_1, r_2] \) if \( r_2 \in P[x, t'] \setminus \{x\} \) and \( Q' := Q[s, r_1] \cup R[r_1, r_2] \cup Q[r_2, t'] \) if \( r_2 \in Q[y, t'] \setminus \{y\} \).

By the assumption, there exists a line segment \( L_{Q'} \in \mathcal{L} \) connecting \( x^* \in P[p, q] \) and \( y^* \in Q' \) such that \( x^* \prec_p t' \) and \( P[x^*, t'] \cup Q'[y^*, t'] \cup L_{Q'} \) forms a closed curve separating \( s \) and \( t \), where \( t' \) is the endnode of \( Q' \) on \( P[p, q] \setminus \{p, q\} \). Since \( L_{Q'} \) must intersect with \( Q \), \( L_{Q'} \) is longer than \( L_Q \), which contradicts the choice of \( Q \). This completes the proof.

If there is a hole in \( \mathcal{H}(r_b, r_p) \) that disconnects \( s \) and \( t \), then we are done. Therefore, we may assume that we have a curve \( Q \) from \( s \) to \( t' \) as in Claim 6. Note that since the proof of Claim 6 is constructive, we can find such \( Q \) in polynomial time.

Consider the \( s \)-\( t \) path \( P' \) obtained by concatenating \( Q[s, t'] \) and \( P[t', t] \). Since \( P' \) is an \( s \)-\( t \) path, there exists a line segment \( L' \in \mathcal{L} \) connecting \( p' \in P' \) and \( q' \in P' \), where \( p' \prec_p q' \), such that \( L' \cup P'[p', q'] \) forms a closed cycle separating \( s \) and \( t \). Among such line segments, we choose \( L' \) such that

1. \( q \prec_p q' \) or \( q = q' \), and
2. subject to (a), \( p' \) is maximum with respect to \( \prec_p \).

Note that if there exists no line segment satisfying the condition (a), that is, for any line segment \( L' \), its endnode \( q' \) satisfies \( q' \prec_p q \), then \( P[q, t] \) is strictly contained in \( P'[q', t] \), which contradicts the choice of \( P \).

Since \( q \) is defined as a maximum node with respect to \( \prec_p \), cycle \( P[p, q] \cup L_P \) separates \( p' \) and \( t' \). Furthermore, since \( P[q, t] \) is as long as possible by the choice of \( P \), either \( P[p, q] \cup L_P \) separates \( s \) and \( q' \) or \( q' = q \). Therefore, \( L' \) interacts with either \( P[p, q] \) or \( L_P \).

**Case 1.** Assume that \( q' \neq q \) and \( L' \) intersects with \( P[p, q] \). Let \( x \) be a node which is an intersection of \( L' \) and \( P[p, q] \) (see Fig. 14). By Claim 6, \( P'[p', t'] \cup L'[p', x] \cup P[x, t'] \) does not separate \( s \) and \( t \). By combining this with the fact that \( P'[p', q'] \cup L' \) separates \( s \) and \( t \), we can see that \( P[x, q'] \cup L'[q', x] \) separates \( s \) and \( t \), which contradicts the maximality of \( q \).

**Case 2.** Assume that \( L' \) intersects with \( L_P \). Let \( x \) be a node which is an intersection of \( L' \) and \( L_P \). In this case, \( C' = L'[p', x] \cup L_P[x, q] \cup P'[p', q] \) forms a closed curve. We divide this case into the following two cases.

**Case 2-1.** We consider the case when \( C' \) does not separate \( s \) and \( t \) (see Fig. 15). By Claim 6, \( p' \) and \( p \) are not contained in a common hole in \( \mathcal{H}(r_b, r_p) \). Then, by the property of \( \mathcal{H}(r_b, r_p) \) described in Section VI, \( q \) and \( q' \) are contained in a common hole in \( \mathcal{H}(r_b, r_p) \). This shows that by concatenating \( P[q, q'] \) and the line segment connecting \( q \) and \( q' \), we obtain a closed cycle separating \( s \) and \( t \), which contradicts the maximality of \( q \).

**Case 2-2.** We consider the case when \( C' \) separates \( s \) and \( t \) (see Figs 16 and 17). Since \( C' \) also separates \( s \) and \( p \), \( P[s, p] \) intersects \( P'[p', t'] \) or \( L'[p', x] \). Let \( y \) be the first intersecting point of \( P[s, p] \) and \( P'[p', t'] \cup L'[p', x] \) when we move from \( p \) to \( s \) along \( P[s, p] \). We consider the following two cases separately.

If \( y \) is on \( L'[p', x] \), then \( P[y, q'] \cup L'[y, q'] \) forms a closed cycle separating \( s \) and \( t \) (see Fig. 16). This contradicts the maximality of \( q \) when \( q \prec_p q' \), and it contradicts the

![Fig. 14. In the case 1, \( P[x, q'] \cup L'[q', x] \) separates \( s \) and \( t \).](image1)

![Fig. 15. In the case 2-1, by concatenating \( P[q, q'] \) and the line segment connecting \( q \) and \( q' \), we obtain a closed cycle separating \( s \) and \( t \).](image2)

![Fig. 16. In the case 2-2, \( y \) is on \( L'[p', x] \).](image3)

![Fig. 17. In the case 2-2, \( y \) is on \( P'[p', t'] \).](image4)
minimality of $p$ when $q = q'$.

If $y$ is on $P'[p', t']$, then $P'' := P'[s, y] \cup P'[y, t'] \cup P'[t', t]$ forms an $s$-$t$ path (see Fig. 17). By the minimality of $p$ and by the maximality of $p'$ (see condition (b)), if $P''[p'', q''] \cup L''$ forms a closed curve separating $s$ and $t$ for some line segment $L'' \in L'$ connecting $p''$ and $q''$, where $p'' \prec_{L''} q''$, then $P[q, t]$ is strictly contained in $P''[q'', t]$. This contradicts the choice of $P$.

By repeating this procedure, we can find a hole or an $s$-$t$ path $P$ satisfying the condition. ■